International Journal of Theoretical Physics, Vol. 45, No. 3, March 2006 (© 2006)
DOI: 10.1007/s10773-006-9035-1

Early Universe with Variable Gravitational and Cosmological Constants

C. P. Singh[1]

Received March 2, 2005; accepted January 24, 2006
Published Online: April 27, 2006

Einstein field equations are considered in zero-curvature Robertson–Walker (R–W) cosmology with perfect fluid source and time-dependent gravitational and cosmological "constants." Exact solutions of the field equations are obtained by using the 'gamma-law' equation of state $p = (\gamma - 1)\,\rho$ in which γ varies continuously with cosmological time. The functional form of $\gamma(R)$ is used to analyze a wide range of cosmological solutions at early universe for two phases in cosmic history: *inflationary phase and Radiation-dominated phase*. The corresponding physical interpretations of the cosmological solutions are also discussed.

KEY WORDS: early universe; gravitational and cosmological "constants."

1. INTRODUCTION

In relativistic and observational cosmology, the evolution of the universe is described by Einstein's field equations together with perfect fluid and an equation of state. Einstein's theory of gravity contains two parameters—Newtonian gravitational 'constant' G and cosmological 'constant' $\wedge$. Normally, these are considered as fundamental constants. The gravitational 'constant' G plays the role of a coupling constant between geometry of space and matter content in Einstein's field equation. In an evolving universe, it appears natural to look at this constant as a function of time. Dirac (1937a) and Dicke (1961) have suggested a possible time-varying gravitational constant. The Large Number Hypothesis (LNH) proposed by Dirac (1937b, 1938) leads to a cosmology where G varies with the cosmic time. There have been many extensions of Einstein's theory of gravitation, with time-dependent G, in order to achieve a possible unification of gravitation and elementary particle physics. Canuto *et al.* (1977a) made numerous suggestions

[1] Department of Applied Mathematics, Delhi College of Engineering, Bawana Road, New Delhi 110042, India; e-mail: cpsphd@rediffmail.com.

based on different arguments that G is indeed time dependent. Beesham (1986) has studied the creation with variable G and pointed out the variation of the form $G \propto t^{-1}$, originally proposed by Dirac (1937a). When the universe is required to have expansion from a finite minimum volume the critical density assumption and conservation of the energy–momentum tensor dictate that G increases in a perpetually expanding universe. The possibility of an increasing G has been discussed by Levit (1980).

On the other hand, There exist several interesting physical problems, where the Einstein's field equation for homogeneous, isotropic, and spatially flat cosmological models with and without cosmological 'constant' $\wedge$ with various matter sources, reduce to particular cases of the second-order nonlinear ordinary differential equations. The $\wedge$-term arises naturally in general-relativistic quantum field theory where it is interpreted as the energy density of the vacuum (see Fulling *et al.*, 1974). It is widely believed that the value of $\wedge$ was large during the early stages of evolution and strongly influenced its expansion, whereas its present value is incredibly small. The possibility of $\wedge$ as a function of time has been considered by Canuto *et al.* (1977b). It is worth noting that cosmological models with a time-dependent cosmological constant $\wedge$, based on Einstein's field equation, have been the subjects of numerous papers in recent years. In a complementary approach, many authors (Abdel-Rahman, 1992; Chen and Wu, 1990; Lima and Maia, 1993) have constructed cosmological models of a more phenomenological character, in which specific decay laws are postulated for $\wedge$ within general relativity. Recently, it has been proposed to link the variation of gravitational 'constant' with that of cosmological 'constant' leaving the form of the field equations unchanged and preserving the conservation of the energy–momentum tensor of the matter content. The generalized Einstein's theory of gravitation with time-dependent G and $\wedge$ has been proposed by Lau (1985). The possibility of variable G and $\wedge$ in Einstein's theory has also been studied by Dersarkissian (1985). This relation plays an important role in cosmology. Berman (1991) and Sistero (1991) have considered the Einstein's field equation with perfect fluid and variables $\wedge$ and G for Robertson–Walker (R–W) line element. Kalligas *et al.* (1992) have studied FRW models with variables $\wedge$ and G and discussed the possible connection with power-laws time dependent of G. Abdussattar and Vishwakarma (1997) presented R–W models with variables G and $\wedge$ by admitting a contracted Ricci collineation along the fluid flow vector. Thus, the implications of time-varying $\wedge$ and G are important to study the early evolution of the universe.

Carvalho (1996) studied a spatially homogeneous and isotropic cosmological model of the universe in general relativity by using equation of state $p = (\gamma - 1)\rho$, where the parameter γ, varies with cosmic time. A unified description of early evolution of the universe has been presented by him in which an inflationary period is followed by a radiation-dominated period. His analysis allows one to consider for both G and $\wedge$.

In the present paper, a spatially homogeneous and isotropic R–W line element is considered with variables G and $\wedge$ in general relativity. Our approach is similar to that of Carvalho (1996) but with time-dependent gravitational and cosmological constants. We apply the same gamma-law equation of state in which the parameter γ depends on scale factor R. We study the evolution of the universe as it goes from an inflationary phase to a radiation-dominated phase. The paper is organized as follows: In Section 2 we present the basic field equations governing the models. In Section 3 we discuss the solutions of the field equation for two different early phases: *inflationary and Radiation-dominated*. The main conclusions are given in Section 4.

2. MODEL AND FIELD EQUATIONS

We consider a spatially homogeneous and isotropic R–W line element

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{1}$$

where *R (t)* is the scale factor and $k = -1, 0$ or $+1$ is the curvature parameter for open, flat and closed universe, respectively. The universe is assumed to be filled with distribution of matter represented by energy–momentum tensor of a perfect fluid

$$T_{\mu\upsilon} = -pg_{\mu\upsilon} + (p+\rho)\,u_\mu u_\upsilon, \ (\text{in the unit with } c = 1) \tag{2}$$

where ρ is the energy density of the cosmic matter and p is its pressure. u_ν is the four-velocity vector such that $u_\mu u^\mu = 1$. The field equations are those of Einstein but with time-dependent cosmological and gravitational constants and given by Weinberg (1971)

$$R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = 8\pi G(t)\,T_{\mu\upsilon} + \wedge(t)\,g_{\mu\upsilon} \tag{3}$$

where $R_{\mu\upsilon}$ is the Ricci tensor, $G(t)$ and $\wedge(t)$ being the variable gravitational and cosmological constants. An additional equation relating the time changes of G and $\wedge$ can be obtained by the covariant divergence of Equation (3), taking into account the Bianchi identity, gives

$$(8\pi G T_{\mu\upsilon} + \wedge g_{\mu\upsilon})^{;\upsilon} = 0 \tag{4}$$

Equations (3) and (4) may be considered as the fundamental equations of gravity with G and $\wedge$ coupling parameters. Using comoving coordinates

$$u_\mu = (1, 0, 0, 0) \tag{5}$$

in Equation (2) and with line element (1), Einstein's field equation (3) yields two independent equations

$$3\ddot{R} = -4\pi G\,(t)\,R\left(3p + \rho - \frac{\wedge(t)}{4\pi G(t)}\right) \tag{6}$$

$$3\dot{R}^2 = 8\pi G(t)R^2\left(\rho + \frac{\wedge(t)}{8\pi G(t)}\right) - 3k \tag{7}$$

In uniform cosmology $G = G(t)$ and $\wedge = \wedge(t)$ so that the conservation equation (4) is given by

$$\dot{\wedge} = -8\pi\dot{G}\rho \tag{8}$$

where dot denotes differentiation with respect to t. Equations (6)–(8) are the fundamental equations. The first two Equations (6) and (7) may be written as

$$8\pi G(t)p = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} + \wedge(t) \tag{9}$$

$$8\pi G(t)\rho = \frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} - \wedge(t) \tag{10}$$

Note that Equations (9) and (10) are formally identical to those of usual cosmology (Ray Chaudhari, 1979). Equations (6) and (7) can be rewritten in terms of the Hubble parameter $H = \dot{R}/R$ to give, respectively

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(t)\,(3p + \rho) + \frac{1}{3}\wedge(t) \tag{11}$$

and

$$H^2 = \frac{8\pi}{3}G(t)\rho + \frac{1}{3}\wedge(t) - \frac{k}{R^2} \tag{12}$$

The system of Equations (8), (11), and (12) may be solved by a physical assumption, i.e., equation of state and from additional explicit assumption on H, G (t), and $\wedge(t)$ in terms of t or H which itself depends on cosmic time t. We take the equation of state

$$p = (\gamma - 1)\rho \tag{13}$$

where γ is an adiabatic parameter varying continuously with cosmological time so that in the course of its evolution the universe goes through a transition from an inflationary phase to radiation-dominated phase. Carvalho (1996) assumed the functional form of γ depends on scale factor as

$$\gamma(R) = \frac{4}{3}\frac{A(R/R_*)^2 + (a/2)(R/R_*)^a}{A(R/R_*)^2 + (R/R_*)^a} \tag{14}$$

where A is a constant and a is free parameter related to the power of cosmic time and lies $0 \leq a < 1$. Here, R_* is certain reference value such that if $R \ll R_*$. Inflationary phase of the evolution of the universe is obtained and for $R \gg R_*$, we have a radiation-dominated phase.

Using Equation (13) into (11), we obtain

$$\dot{H} + H^2 = -\frac{4\pi}{3} G(t)\,(3\gamma - 2)\,\rho + \frac{1}{3} \wedge (t) \tag{15}$$

Eliminating ρ between Equations (8) and (15) for zero curvature ($k = 0$), we get

$$\dot{H} + H^2 = \frac{1}{3}\left(\frac{3\gamma}{2} - 1\right) G(t)\frac{\dot{\wedge}}{\dot{G}} + \frac{1}{3} \wedge (t) \tag{16}$$

To solve Equation (16) we rewrite it in the form

$$HH' + \frac{H^2}{R} = \frac{1}{3}\left(\frac{3\gamma}{2} - 1\right) \frac{G(t)}{R}\frac{\wedge'}{G'} + \frac{1}{3}\frac{\wedge(t)}{R} \tag{17}$$

where a prime denotes differentiation with respect to R.

3. SOLUTION OF THE FIELD EQUATION

Equation (17), involving H, $\wedge(t)$, and $G(t)$ admits solution for H only if $\wedge(t)$ and $G(t)$ are specified. According to the LNH (Dirac, 1937a, 1938), gravitational constant G varies linearly with Hubble parameter, i.e., G decreases with the age of the universe. Chen and Wu (1990) built a dimensional argument to justify $\wedge \propto R^{-2}$. Later on, Lima and Carvalho (1994) argued in favor of a new term of the type $\wedge \propto H^2$. Thus, the phenomenological approach to investigate the cosmological constant is generalized to include a term proportional to H^2 on time dependent of $\wedge$. We obtain the solution of Equation (17) by taking certain assumptions on G and $\wedge$.

3.1. Case (i)

We assume that

$$G(t) = \alpha H \tag{18}$$

and

$$\wedge(t) = \beta H^2 \tag{19}$$

where α and β are dimensionless positive constants.

Substituting the valves of $G(t)$ and $\wedge(t)$ from Equations (18) and (19) into (17), we obtain

$$H' + \left[\frac{\beta+3}{3} - \beta\gamma\right]\frac{H}{R} = 0 \tag{20}$$

Using Equation (14) into (20) and integrating, we get

$$H = \frac{C}{R^{(\beta+3)/3}\left[A(R/R_*)^2 + (R/R_*)^a\right]^{-2\beta/3}} \tag{21}$$

where C is the integration constant. If $H = H_*$ for $R = R_*$, we have a relation between constant A and C, is given by

$$C = H_* R_*^{(\beta+3)/3}\,(1+A)^{-2\beta/3} \tag{22}$$

By use of Equation (22) into (21) and integrating to obtain an expression for t in terms of the scale factor R, is given by

$$H_* R_*^{(\beta+3)/3}\,(1+A)^{-2\beta/3}\,t = \int R^{\beta/3}\left[A(R/R_*)^2 + (R/R_*)^a\right]^{-2\beta/3} dR \tag{23}$$

By defining $q = -(R\ddot{R}/\dot{R}^2)$, it follows from Equation (20) that during the course of evolution the deceleration parameter is given by

$$q = \beta(1-3\gamma)/3 \tag{24}$$

which clearly depends upon R via γ. We solve Equation (23) for two different early phases-inflationary and radiation-dominated.

For inflationary phase ($R \ll R_*$), the second term on right hand side of integral in Equation (23) dominates which gives the solution for scale factor R ($a \neq 0$) as

$$R = R_*\left[\frac{(1-2a)\beta+3}{3}\,\frac{H_*}{(1+A)^{2\beta/3}}\,t\right]^{3/[(1-2a)\beta+3]} \tag{25}$$

Equation (25) shows that during inflation, the dimensions of the universe increase according to law

$$R \propto t^{3/[(1-2a)\beta+3]} \tag{26}$$

which is the case of power-law inflation. If $\beta = 0$, we see that the radius of curvature increases linearly with the age of universe. Now to illustrate some observational predictions of the cosmological model we compute the Hubble parameter, gravitational constant, cosmological constant, and energy density. From Equation (25), we find the following solutions for physical parameters:

$$H = \frac{3}{[(1-2a)\beta+3]}\,t^{-1} \tag{27}$$

$$G = \frac{3\alpha}{[(1-2a)\beta+3]}t^{-1} \tag{28}$$

$$\wedge = \frac{9\beta}{[(1-2a)\beta+3]^2}t^{-2} \tag{29}$$

and

$$\rho = \frac{3(3-\beta)}{8\pi\alpha\,[(1-2a)\beta+3]}t^{-1} \tag{30}$$

For energy density to be positive definite, we must have $\beta < 3$. The energy density tends to infinity as t tends to zero. It is also observed that the spatial volume is zero at $t = 0$. The expanding universe has singularity at $t = 0$. The energy density decreases as time increases and it tends to zero as t tends to infinity. The gravitational ‘constant’ G varies inversely as the age of universe, whereas cosmological ‘constant’ varies inversely as the square of the age of universe, which matches with their natural dimensions. Putting the limiting value $\gamma = 2a/3$ in Equation (24), the asymptotic value of deceleration parameter in the limit $R/R_* \ll 1$, is given by

$$q = (1-2a)\beta/3 \tag{31}$$

For radiation-dominated phase ($R \gg R_*$), the first term on right-hand side of the integral in Equation (23) dominates which gives the solution for scale factor

$$R = R_*\left[(1-\beta)\left(\frac{A}{1-A}\right)^{2\beta/3} H_* t\right]^{1/(1-\beta)} \tag{32}$$

From Equation (32) we observe that during radiation-dominated phase the dimension of the universe increase according to the law

$$R \propto t^{1/(1-\beta)} \tag{33}$$

If $\beta = 0$, the radius of the universe increases linearly with cosmic time. Thus, the volume grows linearly with cosmic time. From Equation (32) we find the following solutions for Hubble parameter, gravitational constant, cosmological constant and energy density

$$H = \frac{1}{(1-\beta)}t^{-1} \tag{34}$$

$$G = \frac{\alpha}{(1-\beta)}t^{-1} \tag{35}$$

$$\wedge = \frac{\beta}{(1-\beta)^2}t^{-2} \tag{36}$$

and

$$\rho = \frac{(3-\beta)}{8\pi\alpha(1-\beta)} t^{-1} \tag{37}$$

respectively. For energy density to be positive, we must have $0 < \beta < 3$. The spatial volume is zero at $t = 0$ and energy density tends to infinity as t tends to zero. Thus, the expanding model has singularity at $t = 0$. Also, energy density decreases with the cosmic time and tends to zero as t tends to infinity. The gravitational constant varies inversely with the cosmic time, whereas $\wedge \propto t^{-2}$. Putting the limiting value $\gamma = 4/3$ in Equation (24) the asymptotic value of deceleration parameter in the limit $R/R_* > 1$, is given by

$$q = -\beta \tag{38}$$

3.2. Case (ii)

We assume that

$$G = \frac{\alpha_1}{H} \tag{39}$$

where α_1 is a positive constant

Using Equations (39) and (19) into Equation (17), we obtain

$$H' + [(1-\beta) + \gamma\beta] \frac{H}{R} = 0 \tag{40}$$

which on integration, gives

$$H = \frac{C_1}{R^{(1-\beta)} \left[A(R/R_*)^2 + (R/R_*)^a\right]^{2\beta/3}} \tag{41}$$

where C_1 is the integration constant. An expression for t in terms of scale factor R is given by

$$H_* R_* (1+A)^{2\beta/3}\, t = \int R^{-\beta} \left[A(R/R_*)^2 + (R/R_*)^a\right]^{2\beta/3} dR \tag{42}$$

Using Equation (42), we obtain the solutions for two different early phases of the universe: inflationary and radiation-dominated.

For inflationary phase we find the following solutions respectively,

$$R \propto t^{3/[2a\beta+3(1-\beta)]} \tag{43}$$

$$H \propto t^{-1} \tag{44}$$

$$G \propto t \tag{45}$$

$$\wedge \propto t^{-2} \tag{46}$$

$$\rho \propto t^{-3} \tag{47}$$

And for radiation-dominated phase we obtain the following solutions respectively,

$$R \propto t^{3/(3+\beta)} \tag{48}$$

$$H \propto t^{-1} \tag{49}$$

$$G \propto t \tag{50}$$

$$\wedge \propto t^{-2} \tag{51}$$

$$\rho \propto t^{-3} \tag{52}$$

From the above solutions we observe that gravitational "constant" increases with the cosmic time in both phases where as cosmological 'constant' varies inversely as the square of the cosmic time. The energy density varies inversely as the cube of the cosmic time and hence tends to infinity as t tends to zero. The expanding model has singularity at $t = 0$.

4. CONCLUSION

We have obtained the cosmological solutions in a spatially homogenous and isotropic R–W line element with varying gravitational and cosmological 'constants'. The "constants" G and $\wedge$ are allowed to depend on the cosmic time t. A unified description of early universe has been presented by using varying gamma of gamma-law equation of state in which inflationary phase is followed by a radiation-dominated phase. In case (i) it has been observed that gravitational constant and energy density vary inversely with the cosmic time where as $\wedge \propto t^{-2}$. The physical parameters have the values in $0 < \beta < 3$. In case (ii) we have obtained that $G \propto t$ and $\wedge \propto t^{-2}$, whereas energy density varies inversely as the cube of the cosmic time for both inflationary and radiation-dominated phases. The results show that the possibility of increasing G has been observed by assuming $G \propto H^{-1}$. The expanding universe has singularity at $t = 0$. In this way, a unified description of early evolution of the universe is possible with variables gravitational and cosmological " constants."

REFERENCES

Abdel-Rahman, A. M. M. (1992). *Physical Review D* **45**, 3497.
Abdusstar and Vishwakarma, R. G. (1997). *Australian Journal of Physics* **50**, 893.
Beesham, A. (1986). *International Journal of Theoretical Physics* **25**, 1295.
Berman, M. S. (1991). *General Relativity and Gravitation* **23**, 465.
Canuto, V., Adams, R. H., Hsieh, S. H., and Tsiang, E. (1977a). *Physical Review D*, **16**, 1643.
Canuto, V., Hsich, S. H., and Adams, P. J. (1977b). *Physics Letters* **39**, 429.
Carvalho, J. C. (1996). *International Journal of Theoretical Physics* **35**, 2019.

Chen, W. and Wu, Y. S. (1990). *Physical Review D* **41**, 695.
DerSarkissian, M. (1985). *Nuovo Cimento B* **88**, 29.
Dicke, R. H. (1961). *Nature (London)* **192**, 440.
Dirac, P. A. M. (1937a). *Nature (London)* **139**, 323.
Dirac, P. A. M. (1937b). *Nature (London)* **139**, 1001.
Dirac, P. A. M. (1938). *Proceeding of Royal Society London* **165**, 199.
Fulling, S. A., Parker, L., and Hu, B. L. (1974). *Physical Review D* **10**, 3905.
Kalligas, D., Wesson, P., and Everitt, C. W. F. (1992). *General Relativity and Gravitation* **24**, 351.
Lau, Y. K. (1985). *Austrian Journal of Physics* **38**, 547.
Levit, L. S. (1980). *Letter Nuovo Cimento* **29**, 23.
Lima, J. A. S. and Carvalho, J. C. (1994). *General Relativity and Gravitation* **26**, 909.
Lima, J. A. S. and Maia, J. M. F. (1993). *Modern Physics Letter A* **48**, 591.
Ray Chaudhari, A. K. (1979). *Theoretical Cosmology*, Clarendon Press, Oxford.
Sistero, R. F. (1991). *General Relativity and Gravitation*, **23**, 1265.
Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.